1. [10]	2. [10]	3. [10]	4. [10]	5. [10]	6. [10]
7. [10]	8. [10]	9. [10]	10. [10]	11. [10]	12. [10]
13. [10]	14. [10]	15. [10]	16. [10]	17. [10]	
Total: [150]					

Ma 116 **Final Exam** **December 8, 2008**

Name: *Pipeline Username*:

Check your lecture

☐ A - N.Strigul (10:00a) ☐ B - P.Dubovski (11:00a) ☐ C - P.Dubovski (12:00p)

Closed book and closed notes.

Calculators and cellphones are to be stored out of sight during the exam.

Show all of your work. Answers without supporting work may receive no credit.

Pledge and sign: I pledge my honor that I have abided by the Stevens Honor System

1 [10 pts] Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \ln(n^2+3)}{2+n^5}$ converge? Does the series converge absolutely?

Solution. Since $\ln(n^2+3) \leq \ln(2n^2) = \ln 2 + 2\ln n \leq 3n$, $2+n^5 > n^5$, then

$$\frac{n\ln(n^2+3)}{2+n^5} \leq \frac{3n}{n^5} = \frac{3}{n^4}.$$

The corresponding series $\sum_{n-1}^{\infty} \frac{3}{n^4}$ is convergent. Consequently, in view of the comparison test, the original series is absolutely convergent and, hence, convergent in the usual sense.

2 [10 pts] Does the series $\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{n!}$ converge? If yes, find its sum.

Solution. This series is just Taylor expansion of e^x about zero at $x = -2$. Its convergence follows from the convergence for e^x for all x. Also, it may be proved directly by the use of alternating series test. Its sum is then e^{-2}.

3 [10 pts] Let $f(x) = \frac{1}{x}$. Find the Taylor series about $x_0 = 2$ for $f(x)$ and its interval of convergence.

Solution. 1. Differentiate and find the pattern: $f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$. Then

$$\frac{1}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n.$$

The convergence radius is $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = 2$. At the endpoints $x = 0$ and $x = 4$ the series is divergent.

4 [10 pts] Find all values of x for which the following series converges.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n} + 2^{n+1}} (x-3)^n.$$

Solution. $x_0 = 3$, $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = 2$. At endpoint $x = 1$ the term of the series becomes

$$\frac{2^n}{\sqrt{n} + 2^{n+1}} = \frac{1}{2 + \frac{\sqrt{n}}{2^n}} \to \frac{1}{2} \neq 0 \text{ if } n \to \infty.$$

From the divergence test we see that the series is divergent in both endpoints. Answer: the series converges if $x \in (-1, 5)$.

5[10 pts] Let

$$\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad \mathbf{b} = -\mathbf{i} + 2\mathbf{j}$$

Let $\mathbf{c} = \mathbf{a} + \lambda\mathbf{b}$. Is it possible to find a value of λ such that vector $\mathbf{c}$ becomes perpendicular to vector $\mathbf{v} = \langle 2, -1, 3\rangle$? If yes, find λ.

Solution. $(\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{v} = \mathbf{a} \cdot \mathbf{v} + \lambda\mathbf{b} \cdot \mathbf{v} = 2 - 4\lambda$. The last is equal to 0 if $\lambda = \frac{1}{2}$.

6[10 pts] Lines l_1 and l_2 have directional vectors

$$\mathbf{a} = \langle 1, 2, 3\rangle, \qquad \mathbf{b} = \langle 0, 4, -2\rangle$$

and pass through points $A(1,2,3)$ and $B(2,-2,1)$, respectively. Find the equation of plane p, which goes through point $P(3,5,-2)$ and is parallel to both lines l_1 and l_2.
Solution. $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -16, 2, 4\rangle$. Then the plane is
$-16(x-3) + 2(y-5) + 4(z+2) = 0$, or $8x - y - 2z = 23$.

7[10 pts] Given two curves

$$\mathbf{r_1}(t) = \sqrt{2t^2 + t - 1}\ \mathbf{i} - t\mathbf{j} + (t^2 - 3)\mathbf{k}$$
$$\mathbf{r_2}(t) = 3\cos t\ \mathbf{i} - (t+2)\mathbf{j} + (t^2+1)\mathbf{k}$$

- (a) Find the intersection point P of these two curves
- (b) Find the angle between these curves at point P.

Solution. (a) We change a parameter in one expression and solve the system of three equations:

$$\sqrt{2t^2 + t - 1} = 3\cos s$$
$$-t = -(s+2)$$
$$t^2 - 3 = s^2 + 1$$

The solution is $t = 2$ and $s = 0$. Then the intersection point is $P(3,-2,1)$.
(b) $\mathbf{r}_1'(2) = \langle \frac{3}{2}, -1, 4\rangle$, $\mathbf{r}_2'(0) = \langle 0, -1, 0\rangle$. The angle between these vectors is

$$\theta = \arccos \frac{\mathbf{r}_1'(2) \cdot \mathbf{r}_2'(0)}{|\mathbf{r}_1'(2)|\,|\mathbf{r}_2'(0)|} = \arccos \frac{2}{\sqrt{77}}.$$

8 [10 pts] Given function $f(x,y) = (2x-1)^2\sqrt{1+y}$, find

- (a) df at point $P(2,3)$.
- (b) ∇f at point $P(2,3)$.
- (c) the directional derivative of $f(x,y)$ at $P(2,3)$ towards $Q(-2,0)$.

Solution. $f_x = 2(2x-1)\sqrt{1+y}$, $f_y = \frac{(2x-1)^2}{2\sqrt{1+y}}$. At point $P(2,3)$ we obtain
$f_x(2,3) = 12$, $f_y(2,3) = 9/4$. Then
(a) $df(P) = 12dx + \frac{9}{4}dy$
(b) $\nabla f(P) = \langle 12, \frac{9}{4}\rangle$
(c) $\vec{PQ} = \langle -4, -3\rangle$. Then unit vector $\vec{u} = \langle -\frac{4}{5}, -\frac{3}{5}\rangle$ and
$D_u f(P) = \nabla f(P) \cdot \vec{u} = -\frac{48}{5} - \frac{27}{20} = -\frac{219}{20} = -10.95$

9 [10 pts] (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at point $P(3,2,-1)$ of function

$$e^{2y + xz^3 - 1} = xy^2 z + x + \frac{5}{2}y^2 - y + 2.$$

(b) Find the equation of the tangent plane to the above surface at point $P(3, 2, -1)$.
Solution. $F(x, y, z) = e^{2y+xz^3-1} - xy^2z - x - \frac{5}{2}y^2 + y - 2$.

$$F_x = z^3e^{2y+xz^3-1} - y^2z - 1, \quad F_y = 2e^{2y+xz^3-1} - 2xyz - 5y + 1, \quad F_z = 3xz^2e^{2y+xz^3-1} - xy^2.$$

Then $F_x(P) = -1 + 4 - 1 = 2$, $F_y(P) = 2 + 12 - 10 + 1 = 5$, $F_z = 9 - 12 = -3$.
(a)We obtain

$$z_x(P) = -\frac{F_x}{F_z} = \frac{2}{3}, \qquad z_y(P) = -\frac{F_y}{F_z} = \frac{5}{3}.$$

(b) $2(x - 3) + 5(y - 2) - 3(z + 1) = 0$, or $2x + 5y - 3z = 19$.

10 [10 pts] (a) Find the critical points for the function $f(x, y) = 3x^2 + 6xy + 7y^2 - 2x + 4y + 1$. Which of them are local maxima, minima and saddle points?
(b) Find the maximum and minimum values of the above function in rectangle $D = [0, 2] \times [-1, 0]$.

Solution. (a) $f_x = 6x + 6y - 2 = 0$, $f_y = 6x + 14y + 4 = 0$. This system yields the only critical point $x = \frac{13}{12}$, $y = -\frac{3}{4}$.
Since $f_{xx} = 6$, $f_{yy} = 14$, $f_{xy} = 6$, we obtain $D > 0$ with $f_{xx} > 0$ which implies the minimum value at point $P(\frac{13}{12}, -\frac{3}{4})$.

(b) We evaluate the values of the function on the rectangle border.
At the left side $x = 0$ we obtain $f(0, y) = 7y^2 + 4y + 1$ with the minimum at $P_2(0, -\frac{2}{7})$. This parabola has no local maximum. Its minimum is greater than the absolute minimum at point P and we disregard it.
At the bottom side $y = -1$ we obtain $f(x, -1) = 3x^2 - 8x + 4$. This parabola has no local maximum, too.
At the right side $x = 2$ we obtain $f(2, y) = 7y^2 + 16y + 9$ with no local maximum.
At the upper side $y = 0$ we obtain $f(x, 0) = 3x^2 - 2x + 1$ with also no local maximum.
Since there is no local maxima in the interior of the sides of the rectangle, then the last step is to evaluate the values of $f(x, y)$ in the corners:
$f(0, 0) = 1$, $f(0, -1) = 4$, $f(2, -1) = 0$, $f(2, 0) = 9$.
Answer: minimum value $f(\frac{13}{12}, -\frac{3}{4}) = -\frac{509}{108}$ (there is no need to calculate this fraction); maximum value $f(2, 0) = 9$.

11 [10 pts] Evaluate $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$.

Solution. $I = \int_0^1 dy \int_0^{3y^2} e^{y^3}\,dx = \int_0^1 3y^2e^{y^3}\,dy = 1$.

12 [10 pts] Evaluate the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2}\,dy\,dx.$$

Solution. The domain is just the unit circle centered at origin. In polar coordinates we obtain

$$\int_0^{2\pi} d\theta \int_0^1 \frac{2r}{(1+r^2)^2} dr = 2\pi \int_0^1 \frac{1}{(1+u)^2} du = \pi.$$

13[10 pts] Formulate the limit comparison test.
Solution. If $a_n > 0$, $b_n > 0$ and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then the series $\sum^\infty a_n$ and $\sum^\infty b_n$ show the same convergence properties (either both converge or both diverge).

14[10 pts] Write the formula for the area of parallelogram formed by vectors **a**, **b**.
Solution. $A = |\mathbf{a} \times \mathbf{b}|$.

15[10 pts] Write the formula for $\frac{df}{dt}$ (Chain rule) for function $f(x, y)$ if $x = x(t)$, $y = y(t)$.
Solution. $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

16[10 pts] (bonus) Suppose that the temperature at the point (x, y, z) on the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ is $T = 8x^2 + 4yz - 16z + 400$. Locate the highest temperature on the ellipsoid.

17[10 pts] (bonus) Find the limit of the sequence

$$\lim_{n\to+\infty} \frac{\sqrt{n^4 + 4n + 11} - n^2}{\sin\frac{1}{n}}.$$

Solution. Using first terms of Taylor series, we obtain $\sqrt{1+x} - 1 \approx \frac{x}{2}$ as $x \to 0$. Similarly, $\sin x \approx x$ as $x \to 0$. Then

$$\lim_{n\to+\infty} \frac{\sqrt{n^4 + 4n + 11} - n^2}{\sin\frac{1}{n}} = \lim_{n\to+\infty} \frac{n^2(\sqrt{1 + \frac{4}{n^3}} - 1)}{\frac{1}{n}} = \lim_{n\to+\infty} n^3 \frac{2}{n^3} = 2.$$